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The meniscus on the outside of a small circular cylinder

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The method of matched asymptotic expansions is used to solve the differential equation describing the shape of a meniscus on the outside of a circular cylinder. Since the perturbation quantity is proportional to the cylinder radius, the solution is valid basically for small cylinders. The predicted meniscus height is compared with numerical data to determine the accuracy of the two-term result; the third term is found but does not improve the estimate.

1. Introduction

When a vertical circular cylinder penetrates a large reservoir of liquid, an axisymmetric meniscus forms around the solid. The meniscus meets the cylindrical surface at an angle dependent on the particular combination of liquid, solid and gas, and then approaches a horizontal plane at large distances from the cylinder. The shape of the interface and the height to which the liquid rises on the cylinder are technically important in areas such as wire coating, but the differential equation governing the shape is nonlinear, and consequently only numerical or approximate analytical solutions are available. The approximate methods refer to the analyses of Ferguson (1912) for cylinders of large radii, and of Nicholson (1949) for menisci whose inclinations are everywhere small. White & Tallmadge (1965) were the first to employ the computer to find the profile of the interface, and more comprehensive calculations were later provided by Huh & Scriven (1969).

The aim of the present work is to find a solution of the governing differential equation by the method of matched asymptotic expansions. This technique can be applied here because the governing equation contains a parameter which can be small under some circumstances: essentially, when the cylinder radius is small or, less often, when the density change across the interface is slight. These conditions limit the usefulness of the solution, but are easily within the range of practical interest.

2. Technique

The technique of Van Dyke (1964) for matched asymptotic expansions will be followed in the present analysis. In using his terminology, some confusion may arise with the terms 'outer' and 'inner': the outer solution will refer to the solution in the original (unstretched) variables and will be valid in the region *close* to the cylinder; conversely, the inner solution, obtained by stretching the

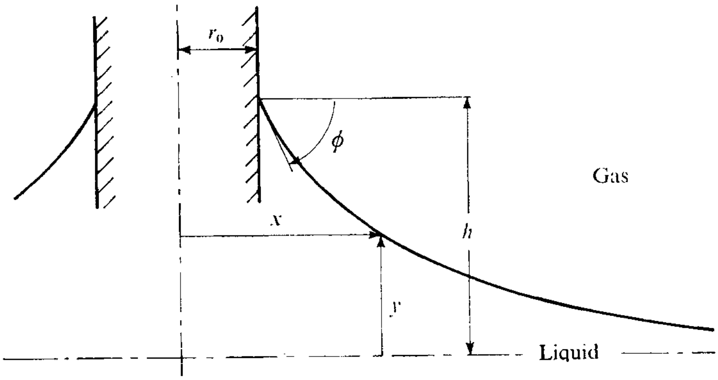

FIGURE 1. Definition sketch for a meniscus attached to the outside of a circular cylinder; $0 \leqslant \phi \leqslant \frac{1}{2}\pi$.

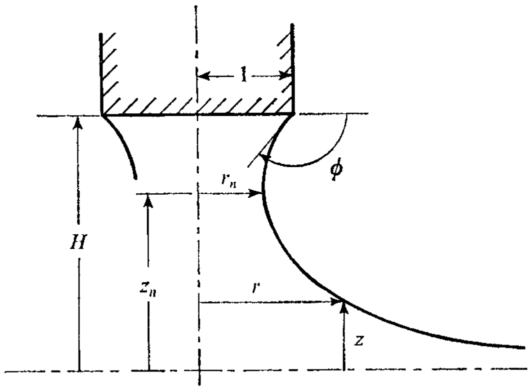

FIGURE 2. Definition sketch for the case $\phi > \frac{1}{2}\pi$, when the meniscus is attached to the bottom of the cylinder.

independent variable, will be uniformly valid *far* from the cylinder. Consequently the adjectives 'outer' and 'inner' are used solely for mathematical purposes in the present problem and do not refer to relative distances, as they do in boundary-layer flows.

The physical situation is sketched in figure 1. A position on the axisymmetric liquid–gas interface is given by either $x(y)$ or $y(x)$, and the liquid meets the solid surface at an angle ϕ, which is 90° minus the contact angle. When the liquid wets the cylindrical surface, as in the sketch, the range of values for ϕ is 0 to $\frac{1}{2}\pi$; the range is extended beyond $\frac{1}{2}\pi$ if the meniscus is attached to the end of the cylinder, as shown in figure 2. Figures 1 and 2 are drawn for positive values of ϕ, but the analysis to be presented also holds for negative values, when the liquid does not wet the solid surface and the meniscus is depressed.

The governing differential equation is based on the fundamental equation of capillarity and has been derived in other work (see Huh & Scriven 1969, for example); when y is the dependent variable, the form of the equation is

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]\left\{\frac{\rho g}{\sigma} y \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} - \frac{1}{x}\frac{dy}{dx}\right\},$$

where ρ is the density of the liquid (or the density difference if the interface separates two liquids) and σ is the interfacial tension. The boundary conditions are (i) $dy/dx = -\tan\phi$ at $x = r_0$ and (ii) $y \to 0$ as $x \to \infty$. When the dimensionless co-ordinates $z \equiv y/r_0$ and $r \equiv x/r_0$ are substituted, the above equation becomes

$$z'' = (1+z'^2)\,[\epsilon^2 z(1+z'^2)^{\frac{1}{2}} - z'/r], \quad \epsilon^2 \equiv \rho g r_0^2/\sigma, \tag{1}$$

where a prime denotes differentiation with respect to the independent variable. The dimensionless parameter ϵ^2 (often called the Bond number) can easily be less than unity, which suggests the application of perturbation techniques to solve the differential equation. Contrary to the usual practice, the inner expansion will be sought first in the work which follows. Proceeding this way simplifies the presentation, but it is necessary to presume for the moment that there is already an outer solution which satisfies the outer boundary condition (i) and which fails to vanish for large r.

2.1. *Inner solution*

To find a solution valid far from the cylinder, the independent variable r is contracted by introducing the inner variable $R = \epsilon r$. Other choices are possible for the inner variable, but only the above form produces a differential equation with a solution which vanishes at infinity. The inner dependent variable $Z(R;\epsilon)$ is not contracted, i.e. $z(r;\epsilon) \equiv Z(R;\epsilon)$, and when R is substituted in (1), the differential equation for Z is

$$Z'' = (1+\epsilon^2 Z'^2)\,[Z(1+\epsilon^2 Z'^2)^{\frac{1}{2}} - Z'/R]. \tag{2}$$

Let the asymptotic expansion for Z be

$$Z(R;\epsilon) = \sum_{n=0}^{\infty} Z_n(R)\,\Delta_n(\epsilon).$$

Since the reduced equation, obtained by formally setting $\epsilon = 0$ in (2), is linear, it is assumed that $\Delta_n = \epsilon^{2n}$. Consequently, the equation for the first term Z_0 is

$$Z_0'' + R^{-1}Z_0' - Z_0 = 0,$$

for which the solutions are the modified Bessel functions of order zero, I_0 and K_0. Since only the latter satisfies the inner boundary condition that $Z \to 0$ as $R \to \infty$, the first term of the inner solution is

$$Z_0 = AK_0(R),$$

where the constant A is to be determined by matching with the outer solution. This solution was obtained earlier by Nicholson (1949) for a meniscus whose slope is everywhere small, and is therefore the solution to be expected in the present problem for the region distant from the cylinder.

2.2. *Outer solution*

The outer solution is assumed to have the form

$$z(r;\epsilon) = \sum_{n=0}^{\infty} z_n(r)\,\delta_n(\epsilon), \tag{3}$$

where the form of the gauge functions $\delta_n(\epsilon)$ cannot be presupposed but must be found by matching with the inner solution. Specifically, the first gauge function is found to be $\delta_0 = \ln\epsilon$ because the one-term expansion of the preceding inner solution is $Z = -A\ln\epsilon$. When $z = z_0 \ln\epsilon$ is substituted into (1), the solution for $z_0(r)$ is a constant, say B, which matches with the inner expansion, if $A = -B$, but which does not satisfy the outer boundary condition, $z'(1) = -\tan\phi$. Other possibilities for δ_0 produce first-term solutions which are more realistic and which satisfy the outer boundary condition, but none matches with the inner solution. Accordingly, the outer boundary condition must be satisfied by the second term.

The expansion of the inner solution to two terms shows that $\delta_1(\epsilon) = O(1) = 1$, and consequently the differential equation for $z_1(r)$ is

$$z_1'' + r^{-1}z_1'(1+z_1'^2) = 0.$$

The solution satisfying $z_1'(1) = -\tan\phi$ is

$$z_1 = D - c\ln\{r+(r^2-c^2)^{\frac{1}{2}}\},$$

where $c = \sin\phi$, D is an undetermined constant, the angle ϕ is restricted to $|\phi| \leqslant \frac{1}{2}\pi$ and $(r^2-c^2)^{\frac{1}{2}}$ means the positive square root of r^2-c^2. This equation describes the shape of a catenoid (Landau & Lifshitz 1959, p. 235), the surface formed by interfacial tension when the pressure difference across the interface is zero. This shape is expected for z_1, for z_1 was obtained from (1) by letting $\epsilon \to 0$, and this procedure is equivalent to eliminating the hydrostatic pressure in the liquid. With z_1 known, the two-term outer solution can be expanded and matched with the following two-term expansion of the inner solution:

$$Z \sim B\ln\epsilon + B(\ln r - \ln 2 + \gamma),$$

where γ is Euler's constant (0·57721...). The matching yields $B = -c$ and $D = c(\ln 4 - \gamma)$, so that the two-term outer solution is

$$z(r;\epsilon) \sim c[-\ln\epsilon + \ln 4 - \gamma - \ln\{r+(r^2-c^2)^{\frac{1}{2}}\}].$$

The maximum height H at $r = 1$ is therefore

$$H \sim \sin\phi[\ln\{4/\epsilon(1+\cos\phi)\} - \gamma] \quad (\epsilon \to 0). \tag{4}$$

Since these results are limited to $|\phi| \leqslant \frac{1}{2}\pi$, the outer solution is reworked in the next section for the case $|\phi| > \frac{1}{2}\pi$.

3. The case $|\phi| > \frac{1}{2}\pi$

The physical situation for $|\phi| > \frac{1}{2}\pi$ differs somewhat from that sketched earlier and is shown in figure 2. In this case, the liquid column has a neck and the radius r_n there is unknown at the outset. Below the neck, the equations describing the profile are the same as those derived before, except that the outer solution must be altered because the outer boundary condition is now applied at the neck, viz. $dz/dr \to \infty$ as $r \to r_n$. When this change is made, the two-term outer solution is

$$z \sim r_n[-\ln\epsilon + \ln 4 - \gamma - \ln\{r+(r^2-r_n^2)^{\frac{1}{2}}\}], \quad z \leqslant z_n.$$

Above the neck, the solution which satisfies the reduced outer equation $z''+r^{-1}z'(1+z'^2) = 0$ and the boundary condition $z'(1) = -\tan\phi$ is

$$z = E + c\ln\{r-(r^2-c^2)^{\frac{1}{2}}\}, \quad z_n \leqslant z \leqslant H,$$

where $(r^2-c^2)^{\frac{1}{2}}$ still refers to the positive root. Fitting the two expressions together at the neck shows that $r_n = c$ and $E = c[\ln(4/\epsilon c^2)-\gamma]$, so that the components of the two-term outer solution for $|\phi| > \frac{1}{2}\pi$ are

$$z \sim \begin{cases} c[\ln(4/\epsilon)-\gamma-\ln\{r+(r^2-c^2)^{\frac{1}{2}}\}], & z \leqslant z_n, \qquad (5a) \\ c[\ln(4/\epsilon c^2)-\gamma+\ln\{r-(r^2-c^2)^{\frac{1}{2}}\}], & z_n \leqslant z \leqslant H. \qquad (5b) \end{cases}$$

The height H can be found from the latter relation and turns out to be the same as that for $|\phi| \leqslant \frac{1}{2}\pi$. Equation (4) therefore holds for all ϕ.

4. Comparison with numerical results

The extensive tables provided by Huh & Scriven (1969) facilitate the comparison of the present work with their numerical results. The comparison is in the form shown in figure 3, in which the values of ϵ and ϕ under a particular curve are those values for which the value of H predicted by (4) differs from the larger

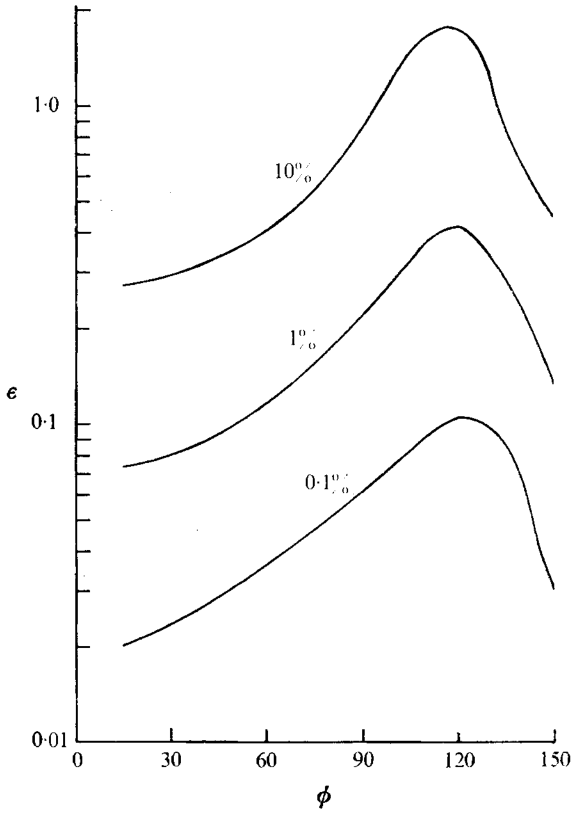

FIGURE 3. The accuracy of equation (4), gauged by the numerical results of Huh & Scriven. The region under a given curve contains the values of ϵ and ϕ for which equation (4) is within the prescribed percentage of the numerical result.

numerical value by less than the prescribed percentage. The figure indicates that the accuracy of the analytical result increases when $\epsilon \to 0$ as expected, and when ϕ is near 120°. This latter behaviour appears to be related to the fact that, while the inner solution is an approximation for a slightly inclined surface, the outer solution is essentially an approximation for large or nearly vertical inclinations. Therefore it is not surprising that the approximation is best when ϕ is around $\frac{1}{2}\pi$, but no explanation can be found for the particular value of 120°.

5. Higher-order terms

A close examination of the analytical and numerical results revealed that the difference was approximately $O(\epsilon)$. This suggested that the next term in the outer expansion might be of that order, and so an effort was made to find additional terms in the asymptotic expansions. This continuation of the analysis, following the method of § 2, was straightforward and hence only a brief account need be given here.

The governing equation for the second term of the inner solution is

$$Z_1'' + R^{-1}Z_1' - Z_1 = c^3K_1^2[\tfrac{3}{2}K_0 + R^{-1}K_1],$$

from which Z_1 can be found by the method of variation of parameters. The solution consists of two integrals which, fortunately, do not need to be evaluated explicitly since only their asymptotic forms are needed for matching with the outer solution. By expanding the integrands for small R (which is equivalent to small ϵ), and integrating term by term, the first few terms of Z_1 are

$$Z_1(R) \sim \tfrac{1}{4}c^3/R^2 - \tfrac{1}{4}c^3 \ln^2 R + (\tfrac{1}{4}c^3 - C)\ln R + O(1), \quad R \to 0,$$

where C is an undetermined constant. The expansion of this solution in the outer variable r shows that $\delta_2 = \epsilon^2 \ln^2 \epsilon$, and so, from (1) and (3), the equation governing the third term of the outer solution is

$$z_2'' + [(1 + 3z_1'^2)/r]\,z_2' = 0.$$

The only solution which satisfies $z_2'(1) = 0$ is $Z_2 =$ constant, and matching with the inner solution shows that this constant is $-\frac{1}{4}c^3$. The three-term outer solution is therefore known, and from it H is found to be

$$H \sim \sin\phi\left[\ln\frac{4}{\epsilon(1+\cos\phi)} - \gamma - \frac{\sin^2\phi}{4}\epsilon^2\ln^2\epsilon\right].$$

A comparison with the numerical data showed that this new estimate for H is not an improvement over (4): the additional term must in fact be *positive* to increase the accuracy. The desired improvement might of course be realized by obtaining still further terms of the outer and inner expansions; this can be done in a fairly direct manner, but it is not clear how many terms would be required.

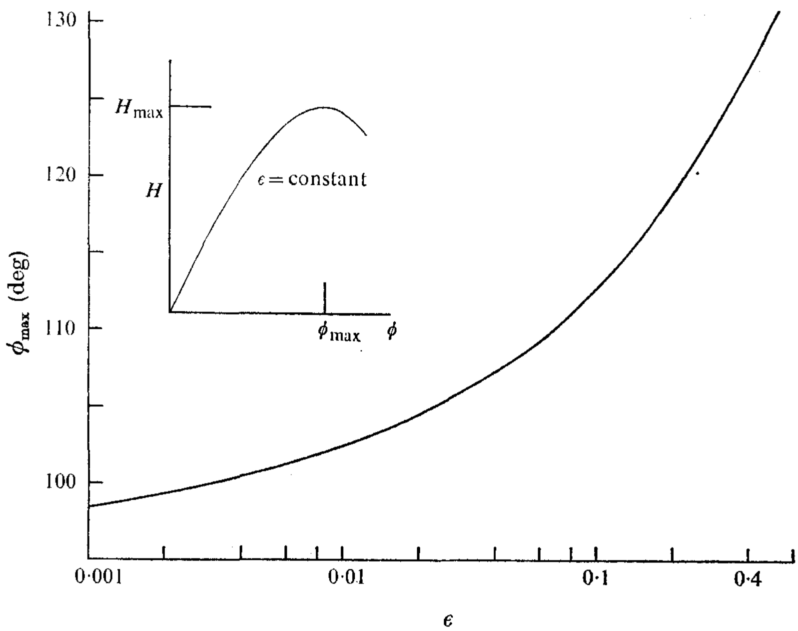

FIGURE 4. The maximum attainable angle ϕ_{max} versus the perturbation parameter ϵ, which represents $(\rho g r_0^2/\sigma)^{\frac{1}{2}}$; this result pertains to the situation sketched in figure 2, where the meniscus is attached to the end of the cylinder.

6. Concluding remarks

The prediction of the height H from (4) is accurate when the parameter $(\rho g r_0^2/\sigma)^{\frac{1}{2}}$ is $O(0{\cdot}1)$ and less. For most liquids this requirement means that the cylinder diameter, or the characteristic dimension in other axisymmetric problems, should be less than about 0·6 mm, a size range applicable to some technological problems. The accuracy of the formula as $\epsilon \to 0$ also suggests that the infinite axisymmetric meniscus can perhaps be used for measurements of interfacial properties.

This paper has concentrated on the result for H, and has presented only regional profiles of the interface. By an additive composite expansion of the outer and inner solutions (Van Dyke 1964), an approximate equation for the entire interface may be obtained which is uniformly valid as $\epsilon \to 0$; in dimensional terms, this equation is

$$y(x) = r_0 \sin\phi\, [\ln 2x - \ln\{x + (x^2 - r_0^2 \sin^2\phi)^{\frac{1}{2}}\} + K_0\{(\rho g/\sigma)^{\frac{1}{2}} x\}],$$

which holds for $|\phi| \leqslant \frac{1}{2}\pi$; when $|\phi| > \frac{1}{2}\pi$, the surface above the neck is given by (5*b*).

Equation (4) reveals that H increases monotonically with ϕ, reaches a maximum at ϕ_{max} say, and then falls off for $\phi > \phi_{max}$ (see the inset in figure 4). Consequently, if H is fixed, (4) admits two possible corresponding values for ϕ. A simple and likely familiar experiment shows, however, that the larger value is physically unrealizeable. If the meniscus is attached to the cylinder end as shown in figure 2, then ϕ can vary by 90°, even though the solid–liquid contact angle remains fixed. Suppose that H and thus ϕ are small initially and the cylinder is then raised slowly: ϕ increases according to (4) until the maximum height is reached, and ϕ equals ϕ_{max}. A further increase causes the meniscus to break away of course, and if the cylinder is lowered, ϕ is observed to return to former values; i.e. ϕ does not exceed ϕ_{max} and attain the theoretical larger values. It is not clear why ϕ cannot go beyond ϕ_{max}, but it appears that this situation is unattainable because of an inherent instability (presumably related to the greater curvature of the interface).

The largest available angle ϕ_{max} is dependent on the physical parameters of the problem, and this relation can be determined by finding the value of ϕ which makes $dH/d\phi = 0$ in (4). This result is illustrated in figure 4 and, like other results in this paper, is uniformly valid as $\epsilon \to 0$.

Note added after review. One of the reviewers referred to a Russian paper which he had been unable to locate but which apparently contained an analytical expression for the meniscus height that is similar to our own. The reference was to Deryagin (1946), who analysed the same problem and whose results are virtually unknown in the literature in the English language. Deryagin did not use matched asymptotic expansions, of course (for this technique was formulated some dozen years later), but he did find approximate solutions close to and far from the cylinder and then, quite remarkably, matched them correctly to yield an expression identical to (4). Hence, our findings are not wholly original, as first thought, but have been available for nearly 30 years; in view of this late discovery, it is necessary to revise the statement of our contribution. In essence, we have formalized the analysis of the problem by the method of matched asymptotic expansions; but doing so, we have shown that Deryagin's result is actually the first term in an asymptotic sequence, and we have indicated how to find higher-order terms. The present method also enabled us to find a single expression for the interface profile which is valid in all regions, i.e. the composite expansion, and to treat the case $\phi > \frac{1}{2}\pi$: these latter results are not obtainable by Deryagin's analysis.

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